Acoustics of Idakkā: An Indian snare drum with definite pitch

Kevin Jose, Anindya Chatterjee, and Anurag Gupta[a)]
Department of Mechanical Engineering, Indian Institute of Technology, Kanpur 208016, India

(Received 10 November 2017; revised 18 March 2018; accepted 26 April 2018; published online 29 May 2018)

The vibration of a homogeneous circular membrane backed by two taut strings is shown to yield several harmonic overtones for a wide range of physical and geometric parameters. Such a membrane is present at each end of the barrel of an idakkā, an Indian snare drum well known for its rich musicality. The audio recordings of the musical drum are analyzed and a case is made for the strong sense of pitch associated with the drum. A computationally inexpensive model of the string-membrane interaction is proposed assuming the strings to be without inertia. The interaction essentially entails wrapping/unwrapping of the string around a curve on the deforming membrane unlike the colliding strings in Western snare drums. The range of parameters for which harmonicity is achieved is examined and is found to be conforming with what is used in actual drum playing and construction. https://doi.org/10.1121/1.5038111

[TS] Pages: 3184–3194

I. INTRODUCTION

A few ingenious drum designs have made it possible to obtain a definite pitch, with several harmonic overtones, out of an otherwise inharmonic vibrating circular membrane. These include timpani, where the kettle is suitably constructed to produce up to four harmonic overtones,[1] and Indian drums, such as tablā, pakhāwaj, and mradangam, where the composite nature of the membrane yields at least ten harmonic modal frequencies.[2–5] Apart from these exceptions, drums are known to possess an indefinite pitch and hence are useful only for generating rhythmic sounds.[6] In this paper, as another exception, we report the acoustics of idakkā, an Indian bi-facial snare drum with a waisted barrel. Idakkā is at present played as a temple instrument in the south Indian state of Kerala and is found frequently in ancient Indian sculptures and musicology texts.[7,8] It is often used to play intricate rāga based melodies and can swiftly move its fundamental in a range of about two octaves. The instrument is worn on the left shoulder and played by striking one of the anterior drumheads with a stick held in the right hand; see Fig. 1. The left hand, while holding the barrel, pushes the waist of the barrel in order to effect the rich tonal variations in idakkā's sound.[9] The purpose of this article is to establish the rich harmonic nature of idakkā first by analyzing the audio recordings and then by proposing a novel numerical model for the string-membrane dynamic interaction.

The two idakkā drumheads are made from a thin hide, of density $\mu \approx 0.1\,\text{kg-m}^{-2}$, obtained from the interior stomach wall of a cow. The thinness of the hide allows for large tension variations in the membrane. The hide is pasted onto a jackfruit wood ring, 2.5 cm thick with an internal diameter of around 20 cm. The two drumheads are connected by a cotton rope which threads through six equidistant holes in the rings and forms a V-shaped pattern between the heads. The shoulder strap has four extensions, each tied to three of the rope segments between the drumheads; see Fig. 1. This arrangement connects all the rope segments symmetrically to the shoulder strap, so that pushing the latter would uniformly change the tension in the drumheads. Four wooden pegs, each about 18 cm long, are inserted between the segments and 16 spherical tassels are attached to each of these pegs (additional illustrations of idakkā are provided in the supplementary material[9]). Whereas the pegs keep the rope segments taut and in proper position, the purpose of hanging the tassels seem to be only of cultural significance.[8]

The barrel is usually made of jackfruit wood, which has a dense fibrous structure with low pore density and hence high elastic modulus. The barrel is around 20 cm in length, with two faces of diameter around 12 cm, and a waist of a slightly lower diameter as shown in Fig. 2 (bottom). The face diameter, being almost half of the drumhead size, ensures that a uniform state of tension indeed prevails within the vibrating membrane even when the drumhead tension is controlled only at six isolated points. The walls of the barrel are about 1 cm thick. The rim has a distinctive convex shape, as seen in Fig. 2 (top left), allowing for the membrane to wrap/unwrap over a finite obstacle during its vibration. This is analogous to vibration of a string over a finite bridge in Indian string instruments leading to a rich spectrum of overtones.[10,11] Two palmyrah fibre strings, of linear density $\lambda \approx 10^{-4}\,\text{kg-m}^{-1}$, are stretched and fixed across each face of the barrel, at about 6 mm from the center, and tied to copper nails on the side; see Fig. 2. Channels are cut into the rim of the barrel to ensure that the fibres sit flush with the rim and therefore with the drumhead. The fibres are soaked in water before they are installed. The tension in the fibres is estimated to be around 6.5 N; the details of this estimation are given in Sec. II A of the supplementary material.[9] The drumheads are held tight against the barrel with the cotton rope and are easily disassembled when not in use. Also, whereas the drumheads can be tuned using the cotton rope, as described in the previous paragraph, the fibres are never tuned. The fibres

[a)]Electronic mail: ag@iitk.ac.in

 0001-4966/2018/143(5)/3184/11/$30.00

FIG. 1. (Color online) An idakkā being played.

are stretched until they are close to the breaking point and then tied onto the rim of the barrel. More details about the structure of idakkā can be found elsewhere.[8,12]

The acoustics of idakkā are governed by several factors, prominent among which are: (i) the string-membrane interaction, (ii) the curved rim of the barrel, (iii) varying membrane tension using the tensioning chords, (iv) the coupling between the two drumheads, and (v) the air loading. The string–membrane interaction in idakkā is tantamount to wrapping-unwrapping of the string around a curve of the vibrating membrane. Such a contact behavior, rather than an impact, is expected due to the higher tension and lower mass density of the palmyrah strings as compared to the metallic strings (used in Western snare drums). The material used for the membrane and the strings, the curved rim, and the intricate method of tensioning the chords are all unique to idakkā. Out of the five factors mentioned above, our emphasis will be to model the string-membrane interaction using a computationally inexpensive model, while assuming the strings to be without mass, without damping, and forming a convex hull below a curve on the vibrating membrane. The influence of the curved rim on the overall acoustics of the drum is discussed briefly in the supplementary material.[9] We will ignore the effects of drumhead coupling and air loading in the present work.

The motivation for our study is provided in Sec. II by analysing audio recordings of the idakkā's sound. A case is made for the rich harmonic sound of the drum. A mathematical model for the string-membrane interaction is proposed in Sec. III. The results of the model are discussed in detail in Sec. IV. These include finding the optimum geometric and material parameters for achieving harmonic overtones, recovering the obtained frequencies from a nonlinear normal mode analysis, and comparing our model with an existing collision model. The paper is concluded in Sec. V.

II. MOTIVATION FOR OUR STUDY

The audio recordings were done on a TASCAM DR-100MKII Linear PCM recorder at a sample rate of 48 kHz with a bit depth of 16. While splicing the audio file into individual samples, the sample rate was changed to 44.1 kHz. The change in the sample rate was a result of saving the spliced files in the default sampling rate of Audacity.[13] We do not anticipate any difference in the results since the frequencies of our interest are on the order of 1 kHz, much smaller than any of these two frequencies. Although the original recordings were done in stereo, we have used only one channel for our analyses. The spectrograms and the power spectral densities (PSD) are plotted using the `spectrogram` and `pwelch` commands, respectively, in MATLAB.[14]

The expert musician (Mr. P. Nanda Kumar) played the seven notes of the musical scale in Indian classical music forward and backward with four strokes of each note. The first stroke, in almost every case, has a swing in frequency in the initial part of the stroke, see Fig. 3 (left), as if the musician is correcting himself to reach the correct pitch as he plays (and hears) the note for the first time. A swing in frequency is also observed towards the end of the last stroke, see Fig. 3 (right), possibly in anticipation of the note to be played next. These swings, in addition to being a corrector or anticipator, are also a reflection of the gamakās related to the rāga being played. It is therefore best to consider either the second or the third stroke of each note for further analysis. We choose the latter. We also note that the spectrograms contain inharmonic content for a small initial time duration; a typical spectrogram is shown in Fig. 4 (left). The inharmonicity is due to the influence from striking of the drum. The harmonic content, however, dominates the spectrogram as well as the auditory experience and sustains itself. We consider only the latter portion of the spectrograms for obtaining the PSD plots. The spectrograms for the case without strings

FIG. 2. (Color online) Distinctive curved shape of the rim of the barrel (top left). A pair of strings installed on an idakkā barrel (top right). Barrel of an idakkā (bottom). A piece of cloth may be used (as shown) to improve the grip. The nails to which snares are tied are also visible close to the edge of the barrel.

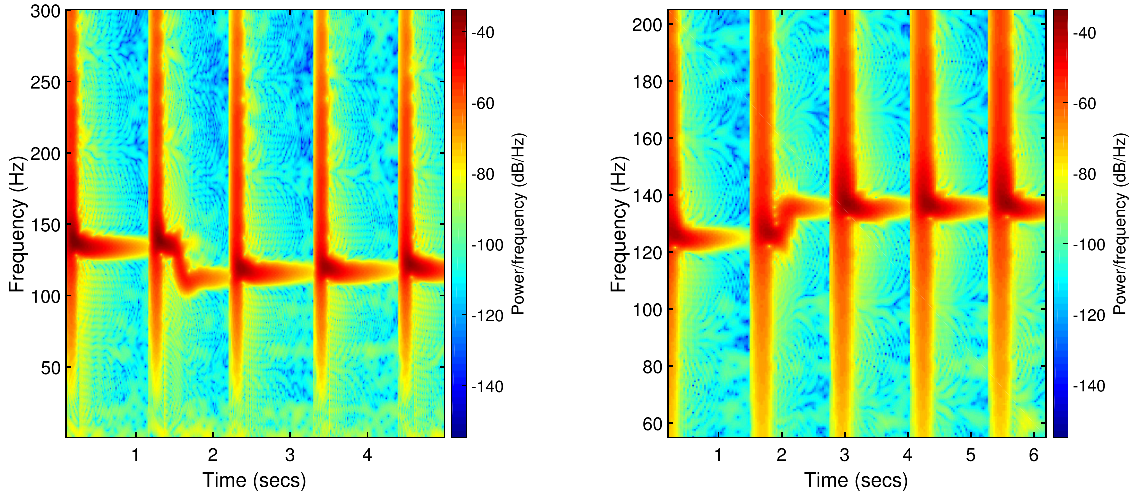

FIG. 3. (Color online) Swings in frequency as seen in the audio recordings as correction (left) and anticipation (right).

show inharmonic content throughout as well as faster decay rates for the higher modes, see for instance Fig. 4 (right).

In Fig. 5, we have three PSD plots each corresponding to the case of idakkā with (top row) and without (bottom row) strings. The fundamental frequency f_0 in each of these plots is different, representing different values of tension in the drumhead. There are three differences between the two cases. First, the power/frequency peaks for overtones are about 20 dB higher in the case with strings compared to the case without strings. Second, the dominant overtones in the PSD plots with strings are always harmonic,[15] whereas there is a significant inharmonic content in the other case. To show this more clearly, we have marked the inharmonic peaks with orange markers in the PSD plots for the case without strings. Some of these inharmonic peaks are present with insignificant amplitudes even in the top row, where the relative importance of harmonic peaks is evident. It should also be noted that, in the top row plots, the third overtone peak is always accompanied by a secondary peak, of much lower intensity, for instance those marked by a blue line at around $3.25f_0$. This is indicative of a beat-like phenomenon. Also, the second harmonic is missing in these plots. This absence is not perceived when listening to an idakkā for well understood psychoacoustical reasons.[16] Third, a harmonic distribution of overtones for the case with strings (top row in Fig. 5) is observed over a large range of membrane tensions. This is noteworthy for a nonlinear problem such as the one present before us. These differences are sufficient to argue in favour of the distinctive role of strings in bringing about a harmonic character to idakkā's sound. The plots without the strings indicate the combined role of the curved rim of the barrel, the air loading, and the drumhead coupling.

We can also calculate the fundamental frequency of the string, seen as an isolated vibrating structure. The tension and the linear density of the string are estimated as 6.5 N and 10^{-4} kg-m^{-1}, respectively. The length of the string is 11 cm. The fundamental frequency can then be calculated as 1159 Hz. Since this value is larger than the frequencies for the fundamental in the PSD plots, in the top row of Fig. 5, we can reasonably argue that the frequencies obtained are a result of the membrane-string interaction and not due to string vibration alone.

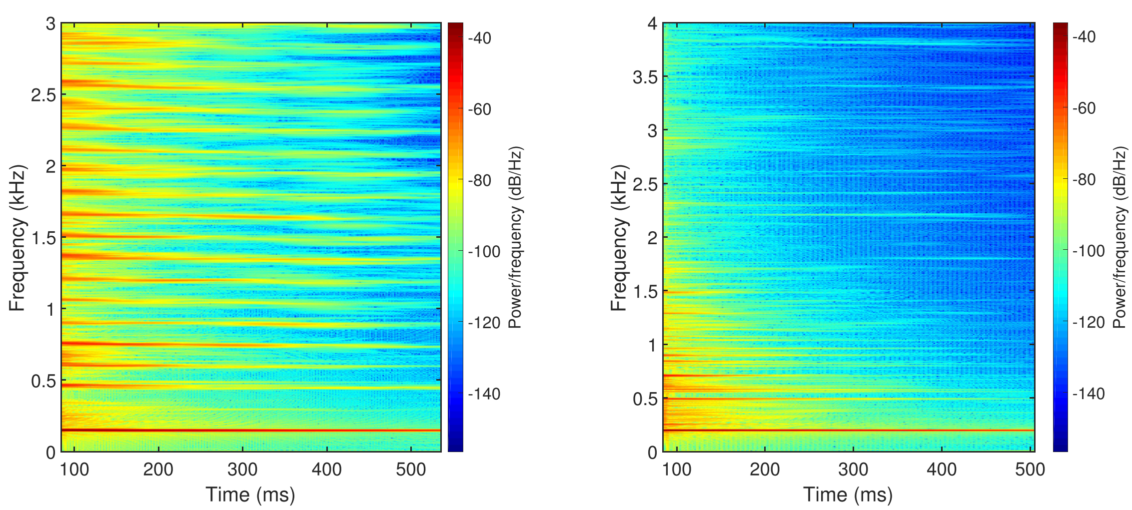

FIG. 4. (Color online) Typical spectrograms with (left) and without (right) the strings.

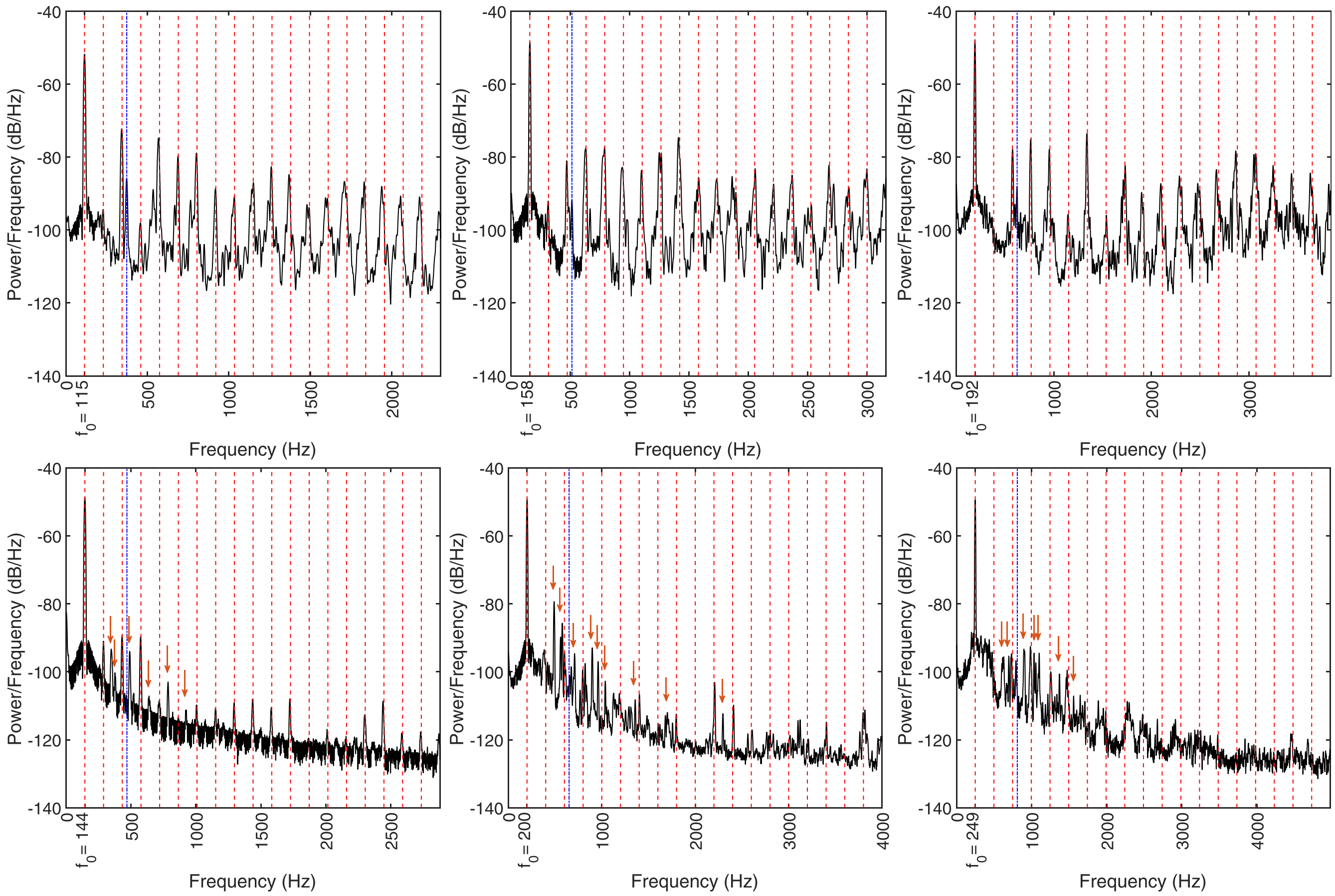

FIG. 5. (Color online) PSD of idakkā drum samples with (top row) and without (bottom row) the strings in idakkā. The dotted lines mark integer multiples of the corresponding fundamental frequency in each plot. The blue line indicates $3.25f_0$. The orange arrows mark the inharmonic peaks.

III. MODEL

As is evident from our analysis of the audio recordings of idakkā, the strings play a central role in bringing about harmonicity in idakkā's frequency spectrum. As a first step towards building a mathematical model for idakkā, we begin by considering the transverse vibration of a uniformly tensed homogeneous circular membrane, with clamped edges, backed by two taut strings. The strings, whose ends are fixed to the circumference of the membrane, are under constant tension and run parallel to each other equidistant from the center of the membrane. At rest, the two strings sit below the plane of the membrane, with negligible distance between the membrane and the strings. This minimalist model is illustrated in Fig. 6(a).

Our model assumes the strings to deform by forming a convex hull around a curve on the deforming membrane, thereby providing a contact force to the vibrating membrane at dynamically varying contact regions; see Figs. 6(b) and 6(c). We neglect the mass of the strings as well as any damping associated with them. The force of string-membrane interaction is therefore determined by statics alone. Furthermore, the strings are assumed to vibrate in a vertical plane orthogonal to the undeformed membrane. The plane of vibration for one of the strings is shown in Fig. 6(b). The deformed position of the strings is derived directly from the shape of the membrane without solving the partial differential equation for the string motion. The deformed string acquires a shape of the convex hull of the curve obtained by intersection of the membrane with the plane. Such a deformation would entail contact of the strings with the membrane at dynamically varying regions. The strings, in this way, are understood to *wrap* around the curves which are the intersection of the deforming membrane with the vertical planes. We compare our model with a penalty-based contact model in Sec. III B. The latter, which solves a coupled system of membrane and string equations of motion, uses a one-sided power law to model the collision while penalising the physically unfeasible inter-penetration. A collision model is necessary for Western snare drums where the inertia of metallic snares cannot be ignored.[17–20]

A. The quasi-static string approximation

The equation of motion of a clamped membrane, of radius R, backed by taut strings is given by

$$\mu W_{tt} = T_M \Delta W - \mu \sigma_{0,M} W_t + \mathcal{F}_S, \tag{1}$$

where $W(r, \theta, t)$ is the transverse displacement of the membrane [(r, θ) are the polar coordinates and t is the time variable], μ is its area density, T_M is the uniform tension per unit length in the membrane, and $\sigma_{0,M}$ is the constant damping coefficient. The subscript t denotes partial derivative with respect to time and Δ is the two-dimensional Laplacian. The force density (per unit length) exerted by the strings on the

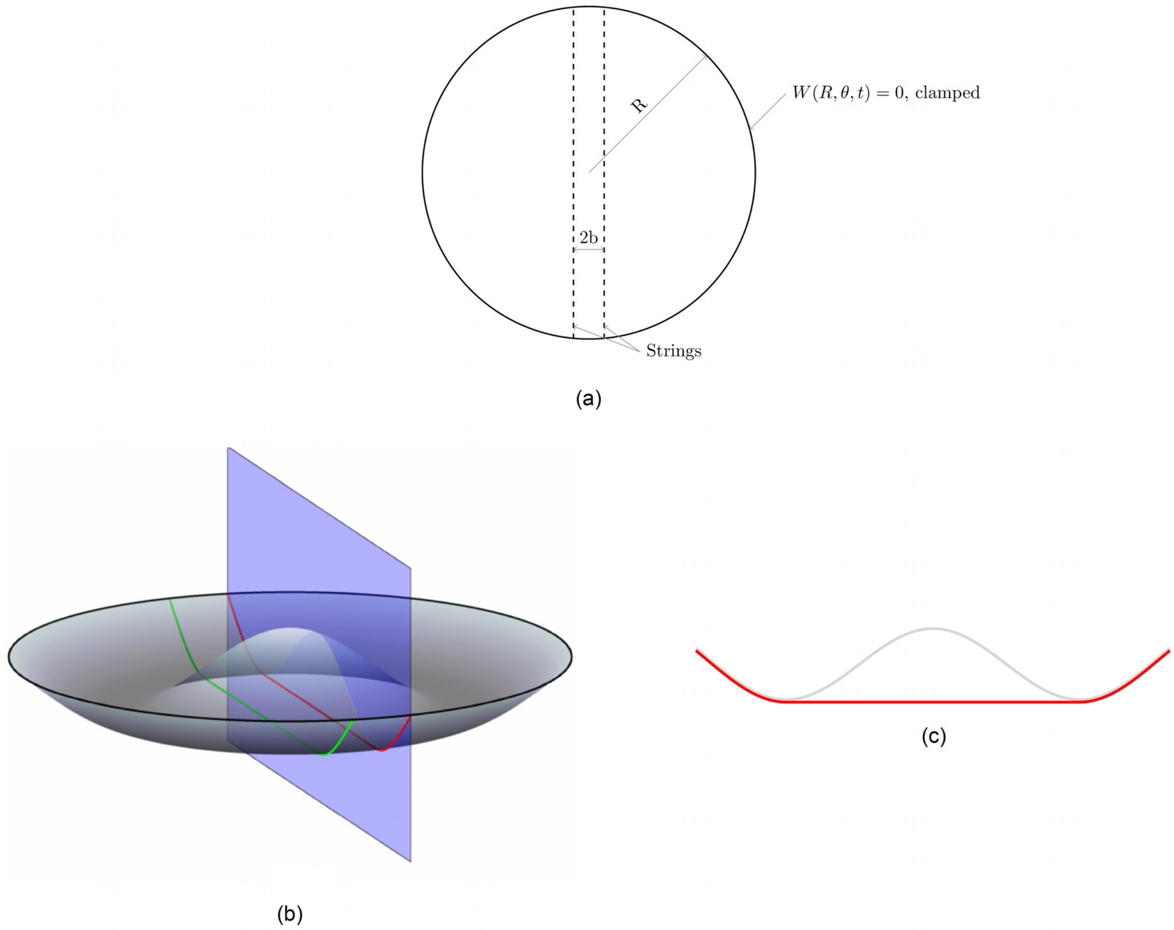

FIG. 6. (Color online) (a) A schematic of the string-membrane configuration. The circular membrane is clamped at the edge and the two strings sit below the membrane. Here, $2b$ is the distance between the strings, R is the radius of the membrane, and $W(r, \theta, t)$ is the transverse displacement of the membrane. (b) An illustration of the strings forming a convex hull around the curve of intersection between the membrane and the plane. A combination of the first two axisymmetrical modes of a uniform circular membrane were used to generate the deformed profile. (c) A sectional view of the string-membrane contact. The plane of section is shown in (b). The grey line indicates the intersection of the membrane with the vertical plane and the red line is the string below the membrane.

membrane is denoted by $\mathcal{F}_S$. The clamped boundary requires $W(R, \theta, t) = 0$. We represent the solutions as $W(r, \theta, t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\eta_{mn}(t)\phi_{mn}(r, \theta)$, where $\eta_{mn}(t)$ are the unknown time-dependent parts of the solution and $\phi_{mn}(r, \theta)$ are the orthonormal mode shapes for a uniform circular membrane of radius R clamped at the edges, i.e., $\phi_{mn}(R, \theta) = 0 \ \forall\ m, n$. Using this expansion, we can convert the partial differential equation [Eq. (1)] into a coupled system of ordinary differential equations

$$\mu\ddot{\eta}_{mn}(t) = -\gamma_{mn}^2 T_M \eta_{mn}(t) - \mu\sigma_{0,M}\dot{\eta}_{mn}(t) + \frac{1}{\pi R^2}\int_A \phi_{mn}\mathcal{F}_S \mathrm{d}A, \quad (2)$$

where $\gamma_{mn} = B_{mn}/R$ with B_{mn} being the nth root of the mth order Bessel function of the first kind. The superposed dot denotes the time derivative with respect to t. The integral is taken over the whole membrane such that dA is the infinitesimal area element.

The membrane experiences a dynamic contact force density $\mathcal{F}_S$ due to its interaction with the two taut strings. We propose that

$$\mathcal{F}_s = -\sum_{i=1}^{2} f^i \delta_L^i, \quad \text{with } f^i = -T_S h^i_{\xi^i \xi^i}, \quad (3)$$

where $h^i(\xi^i, t)$ is the transverse displacement of the string; δ_L^i is the line delta function for the ith string; ξ^i is the intrinsic spatial coordinate on the string; T_S is the uniform tension equal in both the strings; and the subscript ξ^i denotes the partial derivative of the function with respect to the spatial variable. We assume that the strings wrap around the membrane in a sense described above and illustrated in Fig. 6(b). If every point of the membrane remains above the horizontal plane, the strings stays horizontal. Such a consideration lets us ignore the string dynamics making our method computationally less intensive than solving the full coupled problem including the equations of motion for the strings.

We can non-dimensionalize the governing equation [Eq. (2)] and incorporate the force [Eq. (3)] by introducing dimensionless parameters $\tilde{\eta} = \eta/\eta_0$, $\tilde{t} = t/t_0$, $\tilde{h} = h/\eta_0$, and $\tilde{\xi}^i = \xi^i/R$, where $t_0 = R/\sqrt{T_M/\mu}$ and η_0 is any non-zero positive real number with the dimensions of η; we take the magnitude of η_0 to be one. We obtain

$$\ddot{\tilde{\eta}}_{mn} = -B_{mn}^2 \tilde{\eta}_{mn} - \sigma_{0,M} t_0 \dot{\tilde{\eta}}_{mn} + \chi \sum_{i=1}^{2} \left(\int_{-\sqrt{1-(\psi)^2}}^{\sqrt{1-(\psi)^2}} \tilde{h}^i_{\tilde{\xi}^i \tilde{\xi}^i} \phi_{mn}(\tilde{\xi}^i) \mathrm{d}\tilde{\xi}^i \right), \tag{4}$$

where

$$\chi = \frac{T_S}{\pi R T_M} \quad \text{and} \quad \psi = \frac{b}{R}. \tag{5}$$

In writing the last pair of terms in Eq. (4), we have considered the geometry in accordance with Fig. 6(a). In particular, R is the radius of the membrane and $2b$ is the distance between strings. Also, the superposed dot is now indicative of the derivative with respect to variable $\tilde{t}$. The non-dimensionalized governing equation has three dimensionless parameters: ψ is purely geometric in nature, χ is essentially the ratio of string to membrane tensions, and $\sigma_{0,M} t_0$ is the dimensionless damping coefficient term. The tension T_M is the only parameter which can be varied while playing an idakkā. The others are fixed once for all. The integral terms in Eq. (4) are the source of nonlinearity in the equation and also of coupling between the modes.

B. Comparison of the quasi-static string approximation with a penalty-based method

In a penalty-based method, we take the contact force as a one-sided power law

$$\mathcal{F}_s = -\sum_{i=1}^{2} f^i \delta_L^i, \ \text{ with } f^i = -K\left[h^i - W(t, r(\xi^i), \theta(\xi^i))\right]^{\alpha}, \tag{6}$$

where K and α are constants, and $[x] = (1/2)(x + |x|)$.[20] In addition, the equations for string dynamics are also considered in the form

$$\lambda h_{tt}^i = T_S h^i_{\xi^i \xi^i} - \lambda \sigma_{0,S} h_t^i + \lambda \sigma_{1,S} (h^i_{\xi^i \xi^i})_t + f^i, \tag{7}$$

where λ is the linear mass density of the string, and $\sigma_{0,S}$, $\sigma_{1,S}$ are constant damping coefficients. The bending term is ignored due to low bending stiffness of the palmyrah fibres.

For palmyrah fibres in maximal tension, the tension and the power law terms dominate over rest of the terms. This can be seen by writing Eq. (7) in a non-dimensionalized form and considering material values for the palmyrah fibres. The string equation can then be considered in a simplified form

$$0 = T_S h^i_{\xi^i \xi^i} - K\left[h^i - W(t, r(\xi^i), \theta(\xi^i))\right]^{\alpha}, \tag{8}$$

for each string. It is solved numerically using the non-linear solver `fsolve` in MATLAB.[14] We consider only two modes such that $\eta_{01} = -0.1$ mm and $\eta_{02} = 0.1$ mm. We fix $\alpha = 1.3$.[21] The results are shown in Fig. 7. A large value of K is needed to minimize inter-penetration of string and the membrane. We also compare the results with what is used in the quasi-static string approximation. A typical membrane shape ($R = 55$ mm and $b = 6$ mm) is used and the MATLAB function `convhull` is used to obtain the convex hull. We note that for a lower value of K, in Fig. 7(a), there is slight inter-penetration and there is a considerable difference between the contact force as calculated from the penalty method and the quasi-static string approximation. For a higher K value, the inter-penetration reduces significantly and the two contact forces come in better agreement. In a penalty-based method the inter-penetration would never vanish all together unlike the present method where it is zero by construction. Therefore, on one hand, we can view the quasi-static string approximation as a simplification of the penalty based method achieved by ignoring string inertia and damping along with an appropriately high value of K, but on the other hand we note that the former is exact in enforcing avoidance of the inter-penetration of strings into the membrane.

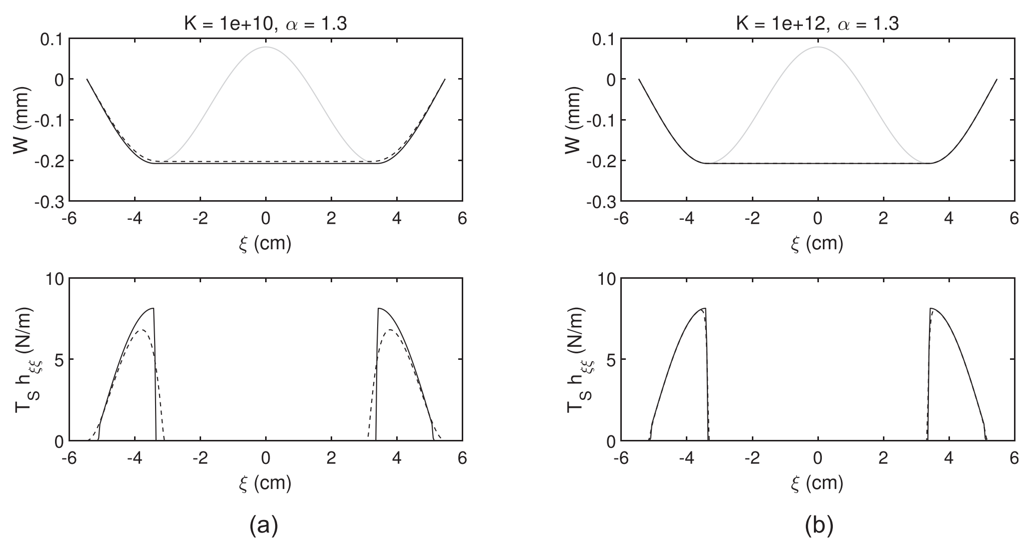

FIG. 7. Comparison of displacements and forces as obtained from Eq. (8) (dotted line) and the quasi-static string approximation (black line) for two different values of K in (a) and (b). A typical membrane shape is shown as a grey line.

IV. RESULTS AND DISCUSSION

We solve Eq. (4) numerically, using `ode113` solver in MATLAB,[14] by considering only the first 12 modes (including odd and even degenerate modes), such that $m = 0, 1, 2, 3, 4$, and $n = 1, 2$. These equations are coupled to each other as well as they are nonlinear in the unknown variables $\tilde{\eta}_{mn}$, both due to the integral term present therein. Indeed, the string displacement $\tilde{h}$ in the integrand is obtained from restricting the convex hull of the membrane (which is in terms of the unknown $\tilde{\eta}_{mn}$) to the one-dimensional domains of the two strings. In doing so, the strings are assumed to move only transversely. The convex hull is calculated using the `convhull` function in MATLAB. The evaluation of string displacements is done concurrently while solving the equation in the sense that the string shape at time t is a function of membrane shape at time t. The integral term in Eq. (4) is approximated using the trapezoidal rule with 201 integration points. For all our simulations, we use an initial condition of a small displacement (away from the strings) of the membrane along the shape of the fundamental mode of uniform membrane vibration. Interestingly, in all our simulations, only four modes, $(0, 1)_e$, $(2, 1)_e$, $(0, 2)_e$, and $(4, 1)_e$ (the subscript e denotes the even mode) remain dominant while the others are excited only weakly. Accordingly, we report the results for only these four modes. Most importantly, for certain parametric values, both significant energy transfer to higher modes and rich harmonicity in the frequency spectrum are observed. This is illustrated in Fig. 8 where the waveform and the PSD plot of the solution for a typical set of parameters are produced. As noted above, it is the interaction of the membrane with the strings which is the source of both non-linearity and coupling between the normal modes. Without the string-membrane interaction terms, as expected, only the $(0, 1)_e$ mode is excited, and the PSD plot shows one isolated peak for the fundamental.

In the following, we begin by looking at the range of membrane tension values, keeping other parameters fixed, for which harmonicity is achieved. This is followed by a similar attempt for the distance between strings. This lets us justify the harmonicity of idakkā over a wide range of tension values on one hand and the optimal design of string placement on the other. Next, we attempt to understand the distinctiveness of idakkā's drumhead design as compared to that of the Western snare drum, knowing well that the latter is bereft of harmonic rich sound.[22] In Sec. IV C, we recover the obtained frequencies by a nonlinear normal mode analysis.

A. Effect of varying χ and ψ

The solutions are obtained by varying T_M over a factor of 3 such that χ is varied in a range of 0.06 to 0.4; see Fig. 9. This represents a change in the fundamental frequency by about 1.7 times. The plots for χ between 0.07 and 0.21 are striking for the appearance of distinct harmonic peaks. The peaks are sharpest for χ around 0.16. The harmonic peaks are also accompanied by smaller peaks of much lower intensity, suggesting a beat-like phenomenon. This was also noted in the spectra obtained from the audio recordings. Outside this range of χ, several inharmonic peaks start to appear, so much so that around $\chi = 0.4$ there is no definite harmonic character in the overtones. The range of desirable χ values may be slightly affected if the effects due to curved rim, air loading, and bi-facial membrane coupling are also incorporated. While obtaining the solution for various values of χ, it must be ensured that the basic assumption of massless strings in the quasi-static string approximation remains justified. In other words, it must be ensured that the inertia term in the string equation remains small, i.e., $1 \ll (T_S t_0^2)/(R^2 T_M) = (\mu T_S)/(\lambda T_M) = (\mu \pi R \chi)/(\lambda)$. The broad range of χ which ensures a near harmonic response is a testament to idakkā's playing over a wide range of membrane tension values. It also ensures that the harmonic response is not too sensitive to the precise tension values in the two strings.

The spectra are also obtained for different values of the geometric parameter $\psi (= b/R)$; see Fig. 10. Other parameters are kept fixed, including $\chi = 0.15$. A variation in ψ represents a variation in both the distance b between the snares and the radius R of the membrane. Sharp harmonic peaks are observed for a wide range of ψ varying between 0 and 0.4. The peaks are the sharpest, with highest intensity, for ψ around 0.1, which is also close to the value usually used in idakkā construction. The secondary peaks are present

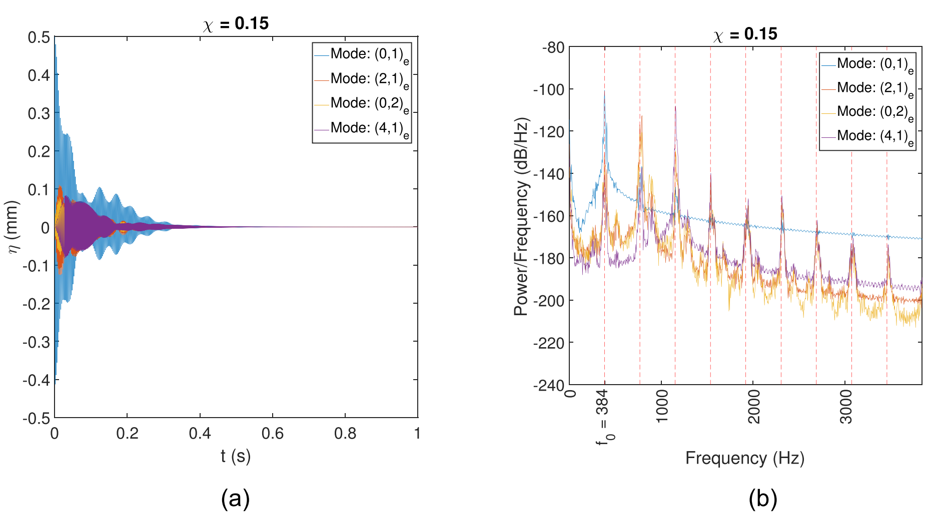

FIG. 8. (Color online) (a) Waveform and (b) PSD plot corresponding to $T_M = 250$ N and $T_S = 6.5$ N (such that $\chi = 0.15$); $\psi = 0.1091$. The dotted lines in (b) mark integer multiples of the fundamental frequency.

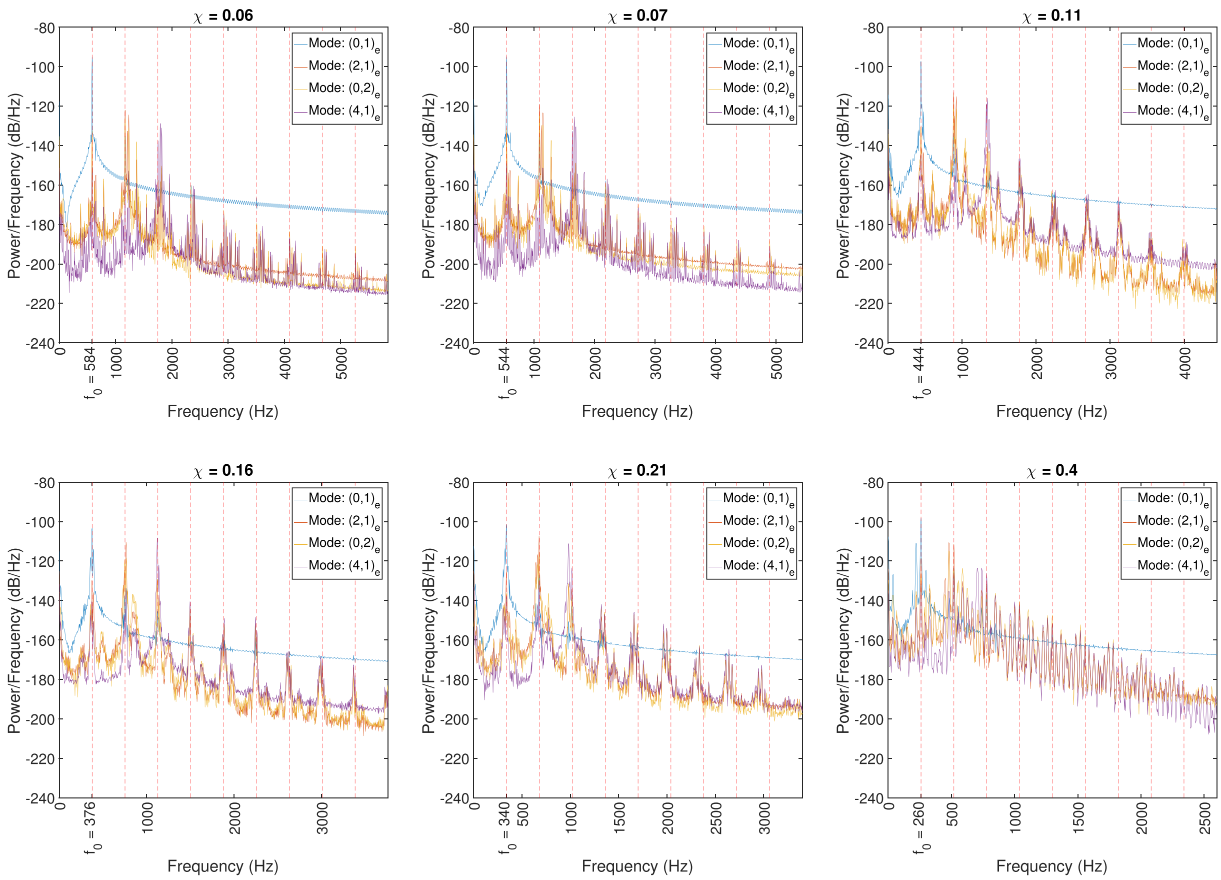

FIG. 9. (Color online) The presence of harmonic overtones for a range of $0.07 < \chi < 0.21$, $\psi = 0.1091$. The dotted lines mark integer multiples of the fundamental frequency.

although with a relatively lower intensity. The robustness of the system in maintaining a near harmonic response over a large range of geometric configurations is in confirmation with the existence of other drum designs, for instance the much smaller udukku, which has only one snare passing through the center (i.e., $\psi = 0$).[8]

B. Comparing idakkā with the snare drum

Unlike idakkā, the Western snare drum is not known to produce harmonic overtones.[17,22] It is then important to understand the differentiating characteristics of the snare action in the former drum. The difference essentially comes from the material of the snare and the tension values in the two drums. Idakkā has natural fibres as snare strings which have a lighter weight when compared to the metallic strings used in snare drums. On the other hand, the tension in idakkā's snares are more than three times that in the snare drum. Additionally, the tension in idakkā's membrane is an order of magnitude lower than in the drumhead of the snare drum. All of this allows us to ignore the string inertia term in the case of idakkā. We use the penalty method to compare the waveform for the two cases. The simulation for the snare drum is performed with geometric and material parameters as given by Torin and Newton.[17] The penalty method-based solutions for idakkā drumhead give results close to those obtained using the quasi-static string approximation, with fundamental frequencies obtained within 2% of each other, as can be observed by comparing Fig. 11 with Fig. 8(b). For want of data, the coefficients to the damping terms for palmyrah fibres are taken equal to that of steel. The material and geometric parameters used are $T_M = 250\,\text{N-m}^{-1}$, $T_S = 6.5\,\text{N}$, $\sigma_{0,M} = 20\,\text{s}^{-1}$, $\sigma_{0,S} = 2\,\text{s}^{-1}$, $\sigma_{1,S} = 0.001\ \text{m}^2\ \text{s}^{-1}$, $\mu = 0.095\,\text{Kg-m}^{-2}$, $\lambda = 1.07 \times 10^{-4}\ \text{Kg-m}^{-1}$, $R = 55\,\text{mm}$, $b = 6\,\text{mm}$, $K = 10^{10}$, and $\alpha = 1.3$. Some of these parameters have been taken from Torin and Newton.[17] The inclusion of string damping is a possible explanation for the non-appearance of higher harmonics in Fig. 11.

The waveforms for the two drum heads are shown in Fig. 12. The waveform in Fig. 12(a) looks different from that in Fig. 8(a) due to incorporation of damping in the former. The waveform of the snare drum shows abrupt changes in the amplitude of the mode shapes. This is caused by the collisions happening at a frequency much lower than the frequency of vibration of the membrane modes, which, in turn, is due to the high mass density and low tension of the metallic snares. On the other hand, no such changes are seen for the idakkā drumhead where the string collisions happen at a frequency higher than that of the membrane vibration. This difference also validates our choice of using the quasi-static method rather than a collision-based method. A second thing to note in these waveforms is the amount of energy transfer that is occurring from the fundamental to the higher modes. It is clear that, in the case of idakkā, several higher modes are predominately excited whereas, for the snare drum, the fundamental dominates over all other modes for all times.

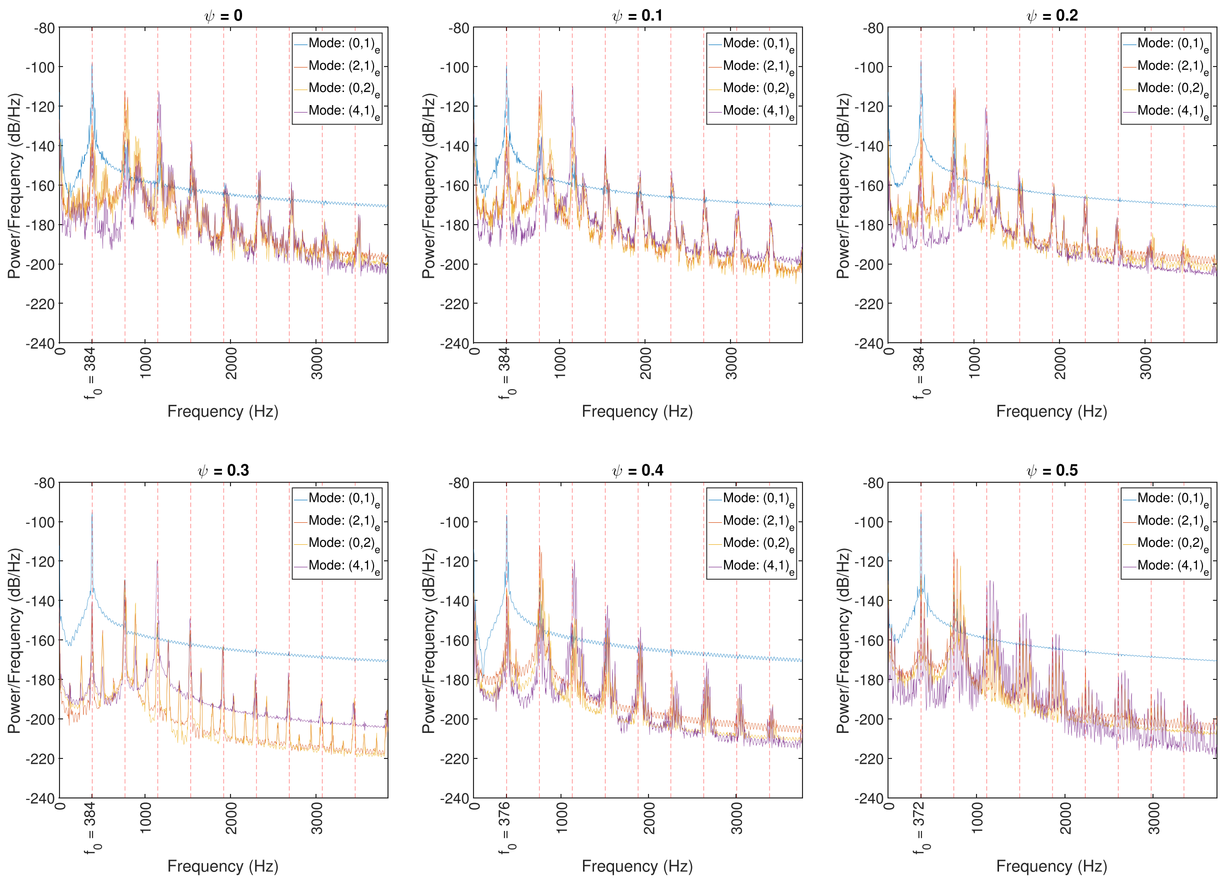

FIG. 10. (Color online) Harmonic response observed for over a range value of $0 < \psi < 0.4$, $\chi = 0.15$. The dotted lines mark integer multiples of the fundamental frequency.

C. Non-linear normal modes

A non-linear normal mode (NNM) of an undamped continuous system is defined as a synchronous periodic oscillation where all the material points of the system reach their extreme values or pass through zero simultaneously, thereby appearing as a closed curve in the configuration space.[23] The purpose of this section is to determine NNMs for a system governed by an undamped form of Eq. (4). We expect the frequencies corresponding to NNMs to be close to those obtained using the full dynamical solution. This will provide an independent validation of the nature of frequency spectra as observed in our simulations. In order to determine NNM, we fix the initial velocity to be zero and look for a time period of the solution such that it crosses the zero velocity condition again in the configurational space. More details are provided in Sec. III of the supplementary material.[9] Fixing parameter values such that $\chi = 0.15$ and $\psi = 0.1091$, we determine four such periodic solutions. The frequencies associated with these solutions are 384.76 Hz (say f), 784.85 Hz ($=2.04f$), 861.19 Hz ($=2.24f$), and 1157.35 Hz ($=3.008f$). The second and the fourth values clearly correspond to the second and the third harmonic, respectively. The third frequency value is close to the neighboring peak to the second harmonic as observed in most of the frequency spectra. The solution corresponding to these four frequencies is shown in Fig. 13. The mode $(0, 1)_e$ is seen to have a mean value greater than zero in all four NNMs. It is possible to find other NNMs by choosing different initial conditions. The four frequencies are also shown superposed on a PSD plot obtained using the quasi-static string approximation in Fig. 14.

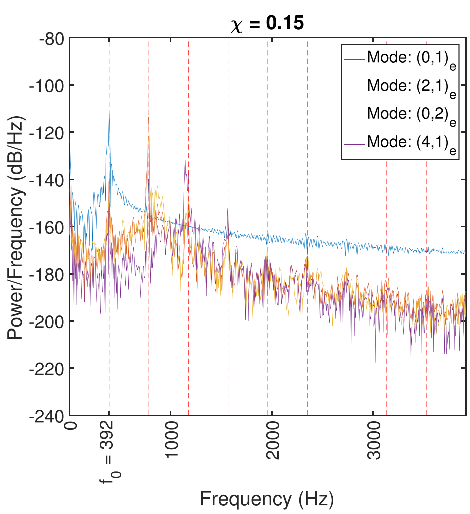

FIG. 11. (Color online) PSD plot obtained for an idakkā drumhead using the penalty method. The dotted lines mark integer multiples of the fundamental frequency.

The frequencies of the NNMs bear emphasis. For wide values of χ, we have already seen that the response of the struck drum is dominated by an almost periodic, but not sinusoidal, response, leading to strong harmonic content. In such cases, however, different NNMs have different

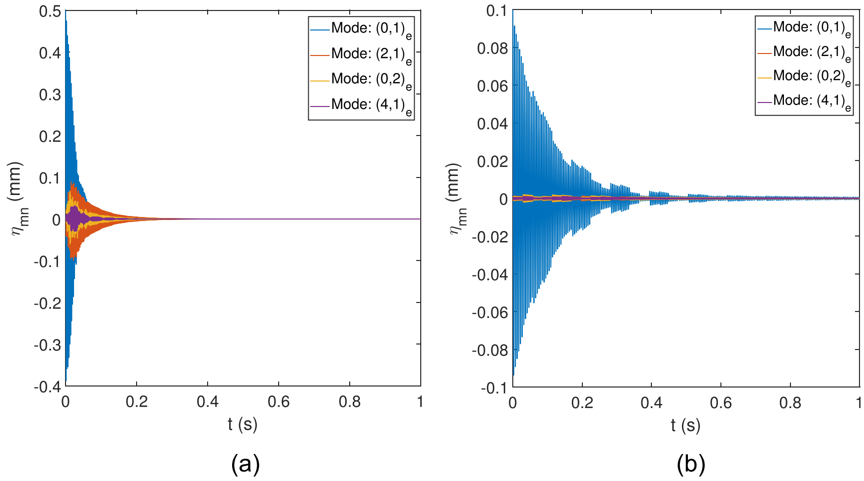

FIG. 12. (Color online) Waveforms obtained from a penalty method-based simulation of (a) an idakkā drumhead and (b) the snare drum drumhead. The physical parameters for the latter is taken from Torin and Newton (Ref. 17).

frequencies which do not bear simple relationships with each other. For the special value of $\chi = 0.15$, however, we find that three *different* NNMs end up with frequencies close to the proportions of 1:2:3, which leads to a particularly strong and rich harmonic response.

The issue can be clarified further as follows. Many conservative nonlinearities might yield several different periodic solutions. Each such periodic solution, if excited in isolation, would be non-sinusoidal and possess harmonics. However, typical nonlinearities make the frequency dependent on amplitude. For the idakkā, the nonlinearity leads to periodic solutions whose frequency does not depend on amplitude. Confining attention to such nonlinearities, which produce periodic solutions whose frequency does not depend on amplitude, in general it may not be possible to easily produce the initial conditions for such periodic solutions. For the case of idakkā, however, our model demonstrates that a typical strike on the drum does produce such initial

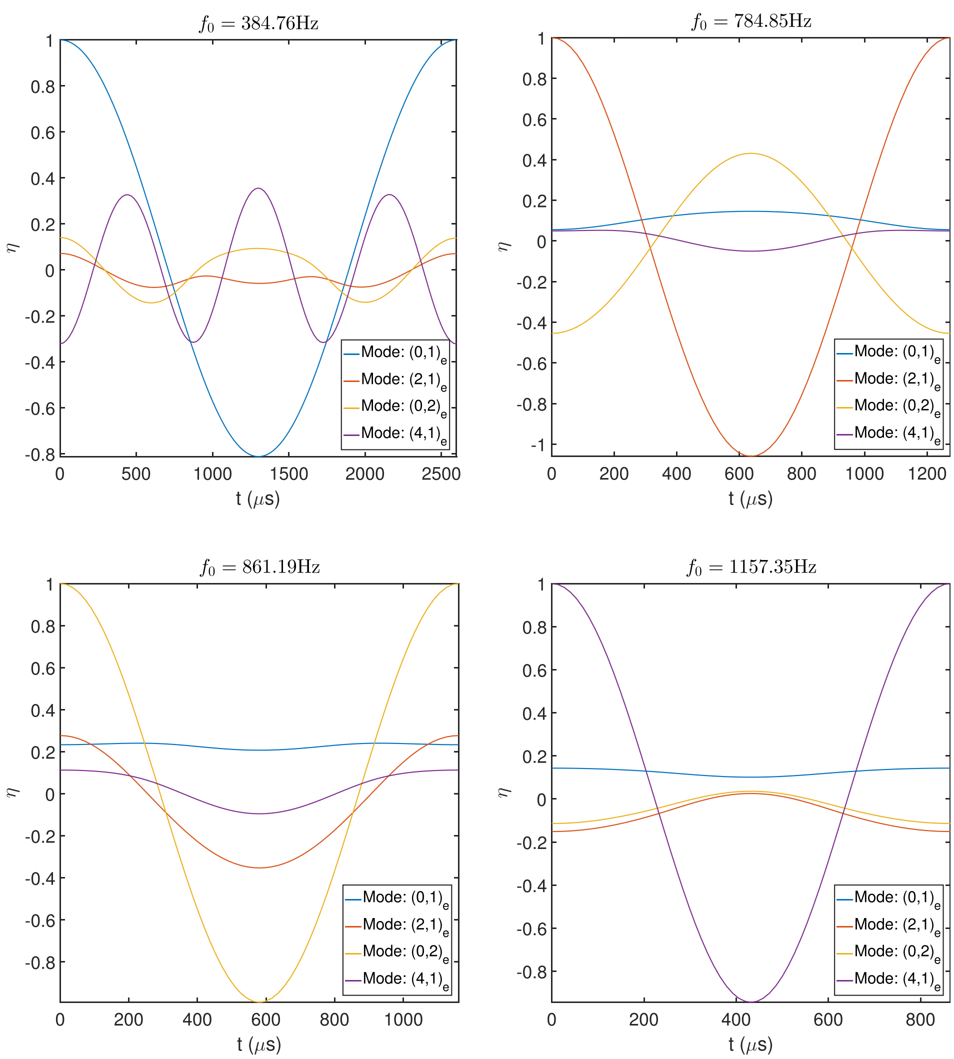

FIG. 13. (Color online) NNM solutions corresponding to the zero velocity initial condition. The variable η on the y-axis denotes a scaled displacement amplitude.

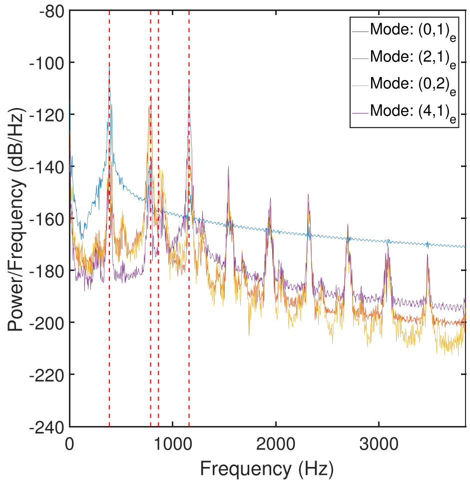

FIG. 14. (Color online) PSD plot of the vibration of a membrane backed by two strings using the quasi-static string approximation superposed with dotted lines which mark the calculated NNM frequencies; $\chi = 0.15$ and $\psi = 0.1091$.

conditions. Finally, even beyond the easy production of such initial conditions, there remains the issue of the existence of several possible NNMs with time periods that do not occur in pleasing proportions. In such cases we expect, as numerics also show for typical χ, that one NNM dominates. The final intriguing, or pleasing, aspect of idakkā seems to be that there is a special value of χ for which three distinct NNMs have frequencies in harmonic proportions. Mathematically, there is in general no reason to expect that multiple NNMs with rationally related frequencies can dynamically coexist in a given solution. However, as our simulations show, this may be occurring to some degree because for the same value of χ, the time response shows a particularly strong and clean harmonic response; see Fig. 14.

V. CONCLUSION

We have investigated idakkā as a musical drum capable of producing a rich spectrum of harmonic overtones. The uniqueness of idakkā is attributed to the nature of the snare action due to the peculiar material of the snares. The other important aspects of idakkā include the curved nature of the barrel rim and the cord tensioning mechanism. The sound of the idakkā is distinctively different from that of Western snare drums, which are otherwise inharmonic,[22] and that of African talking drums and Japanese tsuzumi, both of which have elaborate cord tensioning mechanisms but do not produce a definite pitch. While we have initiated a systematic study into the acoustics of the instrument, several important considerations have been left out for future investigations. These would include modeling the snare action combined with the curved rim, coupled drumheads, and air loading. The interesting mathematical problem of membrane vibration against a unilateral boundary constraint, in the form of a curved rim, also needs further attention.

ACKNOWLEDGMENTS

We are grateful to Mr. P. Nanda Kumar, a professional idakkā player, for helping us in our audio recordings and informing us about various intricacies of the musical instrument.

[1]R. S. Christian, R. E. Davis, A. Tubis, C. A. Anderson, R. I. Mills, and T. D. Rossing, "Effect of air loading on timpani membrane vibrations," J. Acoust. Soc. Am. **76**, 1336–1345 (1984).
[2]C. V. Raman, "The Indian musical drums," Proc. Indian Acad. Sci. **A1**, 179–188 (1934).
[3]B. S. Ramakrishna and M. M. Sondhi, "Vibrations of Indian musical drums regarded as composite membranes," J. Acoust. Soc. Am. **26**, 523–529 (1954).
[4]G. Sathej and R. Adhikari, "The eigenspectra of Indian musical drums," J. Acoust. Soc. Am. **125**, 831–838 (2009).
[5]S. Tiwari and A. Gupta, "Effects of air loading on the acoustics of an Indian musical drum," J. Acoust. Soc. Am. **141**, 2611–2621 (2017).
[6]N. H. Fletcher and T. D. Rossing, *The Physics of Musical Instruments* (Springer, New York, 1998), Chap. 18.
[7]B. C. Deva, *Musical Instruments in Sculpture in Karnataka* (Indian Institute of Advanced Study, Shimla, India, 1989), pp. 33–34.
[8]L. S. Rajagopalan, *Temple Musical Instruments of Kerala* (Sangeet Natak Akademi, New Delhi, India, 2010), Chap. 1.
[9]See supplementary material at https://doi.org/10.1121/1.5038111 to listen to the sample audio. It is a percussive piece in ādi tāla usually played at specific times in the temple. The musician is Mr. P. Nanda Kumar; the supplementary material also contains further details on idakkā design, analysis of audio recordings, and NNMs, and contains a preliminary discussion of the effect of a finite curved rim on membrane vibration.
[10]R. Burridge, J. Kappraff, and C. Morshedi, "The sitar string, a vibrating string with a one-sided inelastic constraint," SIAM J. Appl. Math. **42**, 1231–1251 (1982).
[11]A. K. Mandal and P. Wahi, "Natural frequencies, mode shapes and modal interactions for strings vibrating against an obstacle: Relevance to Sitar and Veena," J. Sound Vib. **338**, 42–59 (2015).
[12]K. Jose, "Vibration of circular membranes backed by taut strings," M. Tech. thesis, IIT Kanpur, India, 2017.
[13]Information available at https://www.audacityteam.org (Last viewed 21 May 2018).
[14]MATLAB. version 9.0 (R2016a). The MathWorks Inc., Natick, Massachusetts, 2016.
[15]The percentage deviation of the dominant peaks is, more precisely, within 1.5% of the nearest integers. The error values are collected in Table I of the supplementary material (Ref. 9).
[16]A. H. Benade, *Fundamentals of Musical Acoustics* (Dover, New York, 1990).
[17]A. Torin and M. Newton, "Collisions in drum membranes: A preliminary study on a simplified system," in *Proceedings of the International Symposium on Musical Acoustics*, Le Mans, France (July 7–12, 2014), pp. 401–406.
[18]A. Torin, B. Hamilton, and S. Bilbao, "An energy conserving finite difference scheme for the simulation of collisions in snare drums," in *Proceedings of the 17th International Conference on Digital Audio Effects*, Erlangen, Germany (September 1–5, 2014), pp. 145–152.
[19]S. Bilbao, A. Torin, and V. Chatziioannou, "Numerical modeling of collisions in musical instruments," Acta Acust. united Ac. **101**, 155–173 (2015).
[20]S. Bilbao, "Time domain simulation and sound synthesis for the snare drum," J. Acoust. Soc. Am. **131**, 914–925 (2012).
[21]A. Torin (private communication, 2017).
[22]T. D. Rossing, I. Bork, H. Zhao, and D. O. Fystrom, "Acoustics of snare drums," J. Acoust. Soc. Am. **92**, 84–94 (1992).
[23]A. F. Vakakis, "Non-linear normal modes (NNMs) and their applications in vibration theory: An overview," Mech. Syst. Signal Process. **11**, 3–22 (1997).